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A NUMERICAL SOLUTION FOR TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND WITH KERNELS OF THE LOGARITHMIC POTENTIAL FORM

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SUMMARY

Two-dimensional Fredholm integral equations with logarithmic potential kernels are numerically solved. The explicit convergence of these solutions to their true solutions is demonstrated. The results are based on a previous work in which numerical solutions were obtained for Fredholm integral equations of the second kind with continuous kernels.

INTRODUCTION

Previously (ref. 1), the convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind was proven. Later on (ref. 2), these results were extended to two dimensions for continuous kernels. However, since a class of physical problems involves kernels of the logarithmic potential form, this study extends the theory to kernels of this type.

MAIN DEVELOPMENT

In a recent report (ref. 2), it was shown that the following system of equations,

$$x(t_{ij}) - \lambda \sum_{k=1}^{n} \sum_{\ell=1}^{n} h(t_{ij};t_{k\ell})x(t_{k\ell})\Delta = y(t_{ij}) \tag{1}$$

converge to the exact solution of the two-dimensional Fredholm integral equation of the second kind,

$$x(r,s) - \lambda \int_0^1 \int_0^1 h(r,s;t_1,t_2) \times (t_1,t_2)dt_1\, dt_2 = y(r,s) \tag{2}$$

when the kernel $h(r,s;t_1,t_2)$ is a continuous function over the unit square $[0,1] \times [0,1]$ and $y(r,s)$ is also a continuous function over the unit square.

Equation (2) will be regarded as a functional equation in the Banach space $X = C^0$ of continuous functions on the unit square ($[0,1] \times [0,1]$) and, typically, will be expressed in the following form:

$$Kx \equiv x - \lambda Hx = y \tag{3}$$

The system (1) is regarded as an approximate functional equation in the space $\bar{X} = R^{\bar{n}}$ and typically is expressed in the following form:

$$\bar{K}\bar{x} \equiv \bar{x} - \lambda\bar{H}\bar{x} = \phi y \tag{4}$$

Let $\tilde{X}$ be a subspace of X. Define the mapping ϕ_0 in $\tilde{X}$ onto $\bar{X}$ as follows: if $\tilde{x} \in \tilde{X}$, then $\phi_0\tilde{x} = \bar{x}$.

The differences between this work and that of reference 2 are due to differences arising in proving the following three conditions:

$$\forall \tilde{x} \in \tilde{X}, \qquad \|\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}\| \le \zeta\|\tilde{x}\| \tag{5}$$

$$\forall x \in X, \qquad \exists\, \tilde{x} \in \tilde{X} \ni \|Hx - \tilde{x}\| \le \zeta_1\|x\| \tag{6}$$

$$\exists\, \tilde{y} \in \tilde{X} \ni \|y - \tilde{y}\| \le \zeta_2\|y\| \tag{7}$$

and in showing that ζ, ζ_1, and ζ_2 go to zero as the mesh size goes to zero.

Conditions (5)-(7) are shown for the case of continuous kernels in reference 2. In this work, for the case of logarithmic kernels,

Let $\bar{H}$ of equation (4) be defined as:

$$\bar{H} \equiv \Delta \begin{bmatrix} 0 & h(t_{11},t_{12}) & \cdots & h(t_{11},t_{1n}) & \cdots & h(t_{11},t_{nn}) \\ h(t_{12},t_{11}) & 0 & & & & h(t_{12},t_{nn}) \\ \vdots & & & & & \vdots \\ h(t_{1n},t_{11}) & & & & \cdots & h(t_{1n},t_{nn}) \\ \vdots & & & & & \vdots \\ h(t_{nn},t_{11}) & h(t_{nn},t_{12}) & \cdots & h(t_{nn},t_{nn-1}) & & 0 \end{bmatrix} \tag{8}$$

Lemma 1: H maps $C^0 \to C^0$.

Proof of Lemma 1: Since H is a bounded linear operator, this result is immediate.

In $\tilde{X}$ (a subspace of X),

$$\tilde{K}\tilde{x} \equiv \tilde{x} - \lambda\tilde{H}\tilde{x} = Py \tag{9}$$

which is the approximate equation as in reference 2, since

$$\phi_0^{-1}\phi = P$$

Equation (9) becomes

$$. - \lambda\tilde{H}\tilde{x} = \phi_0^{-1}\phi y \tag{10}$$

Also,

$$\phi_0(\tilde{x} - \lambda\tilde{H}\tilde{x}) = \phi y \tag{11}$$

$$\Rightarrow \quad \bar{x} - \lambda\bar{H}\bar{y} = \phi y \tag{12}$$

where $\bar{x} = \phi_0\tilde{x}$.

Lemma 2: Condition (5) is satisfied; that is,

$$\forall\tilde{x} \in \tilde{X}\ , \quad ||\phi H\tilde{x} - \bar{H}\phi_0\tilde{x}|| \leq \zeta\,||\tilde{x}|| \tag{13}$$

Proof of Lemma 2: For a given $\epsilon > 0$, $\exists$ an integral operator H_1 with a continuous kernel $\ni$ (see the appendix),

$$||H_1 - H|| < \epsilon \tag{14}$$

Since H_1 is an integral operator with a continuous kernel, the results of reference 2 are applicable. In particular,

$$\exists\,\bar{H}_1 \ni \forall\tilde{x} \in \tilde{X}\ , \quad ||\phi H_1\tilde{x} - \bar{H}_1\phi_0\tilde{x}|| \leq \zeta_1||\tilde{x}|| \tag{15}$$

Hence,

$$||\phi H\tilde{x} - \bar{H}_1\phi_0\tilde{x}|| \leq ||\phi H\tilde{x} - \phi H_1\tilde{x}|| + ||\bar{H}_1\phi_0\tilde{x} - \bar{H}\phi_0\tilde{x}|| + ||\phi H_1\tilde{x} - \bar{H}_1\phi_0\tilde{x}|| \tag{16}$$

$$\leq ||\phi||\ ||\bar{H}_1 - \bar{H}||\ ||\tilde{x}|| + ||\bar{H}_1 - \bar{H}||\ ||\phi_0||\ \ ||\tilde{x}||$$

$$+ ||\phi H_1\tilde{x} \ - \bar{H}_1\phi_0\tilde{x}||$$

$$\leq \zeta||x|| \tag{17}$$

Lemma 3: Condition (6) is satisfied; that is,

$$\forall x \in X\ , \quad \exists \tilde{x} \in \tilde{X} \ni \|Hx - \tilde{x}\| \leq \bar{\zeta}_1 \|x\| \tag{18}$$

Proof of Lemma 3: From the proof of Lemma 2, we know that

$$\forall x \in X\ , \quad \exists \tilde{x} \in \tilde{X} \ni \|H_1 x - \tilde{x}\| \leq \zeta_1 \|x\| \tag{19}$$

Again, from the proof of Lemma 2, $\exists H_1 \ni$ for a given $\epsilon > 0$,

$$\|H_1 - H\| < \epsilon \tag{20}$$

Noting that

$$Hx - \tilde{x} = (Hx - H_1 x) + H_1 x - \tilde{x} \tag{21}$$

Now, applying Schwarz's inequality,

$$\|Hx - \tilde{x}\| \leq \|Hx - H_1 x\| + \|H_1 x - \tilde{x}\| \tag{22}$$

$$\leq \|H - H_1\| \ \|x\| + \|H_1 x - \tilde{x}\| \tag{23}$$

$$\leq \bar{\zeta}_1 \|x\| \tag{24}$$

Condition (7) follows readily from Lemma 3.

From the proofs of Lemmas 2 and 3, it readily follows that ζ, ζ_1, and ζ_2 tend to zero as the mesh size of the partition tends to zero.

CONCLUSION

For a given $\epsilon > 0$, the solution x^* of the Fredholm integral equation $x - \lambda Hx = y$ on $[0,1] \times [0,1]$ with a logarithmic kernel for continuous y can be approximated by a function $\bar{x}^* \ni \|x - \phi_0^{-1}\bar{x}^*\| < \epsilon$; $\bar{x}^*$ satisfies an equation of the form $\bar{x} - \lambda \bar{H}\bar{x} = \phi y$. Hence, the desired accuracy can be achieved by appropriately restricting the mesh size Δ.

APPENDIX

CONSTRUCTION OF APPROXIMATE INTEGRAL OPERATOR H_1

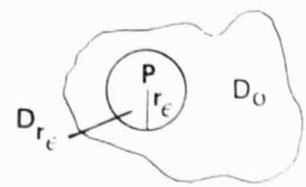
Let H be an integral operator defined as follows:

$$H[\ ] = \int_D \log u_{PQ} [\ ]_Q \, dS_Q$$

Lemma: Given $\epsilon > 0$, $\exists$ an integral operator H_1 with a continuous kernel $\ni \|H_1 - H\| < \epsilon$.

Proof:

$$H[\ ] = \int_D \log r_{PQ} [\ ]_Q \, dS_Q$$



$$D = D_0 \cup D_{r_\epsilon}$$

$$r = |\bar{P} - \bar{Q}|$$

$$= \int_{D_0} \log r[\ ]_Q \, dS_Q + \int_{D_{r_\epsilon}} \log r[\ ]_Q \, dS_Q$$

In D_0, define

$$H_1[\ ] = \int_{D_0} \log r[\ ] \, dS$$

In D_{r_ϵ},

$$H[\ ] = \int_{D_{r_\epsilon}} \log r[\ ] \, dS$$

In D_{r_ε}, define

$$H_1[\ ] = \int_{D_{r_\varepsilon}} \log r_\varepsilon [\ ]\, dS$$

$$= \log r_\varepsilon \int_{D_{r_\varepsilon}} [\ ]\, dS$$

$$\therefore\ H[\ ] - H_1[\ ] = \int_{D_{r_\varepsilon}} \log r[\ ]\, dS_Q - \int_{D_{r_\varepsilon}} \log r_\varepsilon [\ ]\, dS_Q$$

$$= \int_{D_{r_\varepsilon}} [\log r - \log r_\varepsilon]\, [\ ]\, dS_Q$$

$$\|(H - H_1)[\ ]\| \le \|[\ ]\| \qquad \int_{D_{r_\varepsilon}} \log r\, dS_Q - \int_{D_{r_\varepsilon}} \log r_\varepsilon\, dS_Q$$

$$\int_{D_{r_\varepsilon}} \log r_\varepsilon\, dS = (\log r_\varepsilon)\, \frac{r_\varepsilon^{\ 2}}{2}$$

$$|\log r - \log r_\varepsilon|\, dS \le \int \log r_\varepsilon\, dS$$

$$\le \frac{r_\varepsilon^{\ 2}}{2} \log r_\varepsilon$$

$$\therefore\ \|H - H_1\| \le \frac{r_\varepsilon^{\ 2}}{2} \log r_\varepsilon$$

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